Chapter 6 solution

6.1 (a).

$$F_{U_1}(u) = \int_0^{\frac{u+1}{2}} 2(1-y)\,dy$$
$$= u+1-\frac{(u+1)^2}{4}, \quad -1 \le u \le 1$$

$$\Rightarrow F_{U_1}(u) = \begin{cases} 0, & u < -1 \\ u+1-\frac{(u+1)^2}{4}, & -1 \le u \le 1 \\ 1, & u > 1 \end{cases}$$

$$\Rightarrow f_{U_1}(u) = \frac{1-u}{2}, \quad -1 \le u \le 1$$

(b)

$$F_{U_2}(u) = \begin{cases} 0 & u < -1 \\ u+\frac{(1-u)^2}{4}, & -1 \le u \le 1 \\ 1 & u > 1 \end{cases}$$

$$\Rightarrow f_{U_2}(u) = \frac{1+u}{2}, \quad -1 \le u \le 1$$

(c)

$$F_{U_3}(u) = \begin{cases} 0, & u < 0 \\ 2\sqrt{u} - u, & 0 \le u \le 1 \\ 1, & u > 1 \end{cases}$$

$$\Rightarrow f_{U_3}(u) = \frac{1}{\sqrt{u}} - 1, \quad 0 \le u \le 1$$

(d) $E(U_1) = -\frac{1}{3}$

$E(U_2) = \frac{1}{3}$

$E(U_3) = \frac{1}{6}$

(e). $E(U_1) = E(2Y-1) = -\frac{1}{3}$

$E(U_2) = E(1-2Y) = \frac{1}{3}$

$E(U_3) = E(Y^2) = \frac{1}{6}$

6.2 (a)

$$F_{U_1}(u) = \begin{cases} 0, & u < -3 \\ \frac{1}{2} \times \frac{u^3}{27} + \frac{1}{2}, & -3 \le u \le 3 \\ 1 & u > 3 \end{cases}$$

$$\Rightarrow f_{U_1}(u) = \frac{u^2}{18}, \quad -3 \le u \le 3$$

(b)

$$F_{U_2}(u) = \begin{cases} 0, & u < 2 \\ \frac{1}{2} - \frac{1}{2}(3-u)^3, & 2 \le u \le 4 \\ 1, & u > 4 \end{cases}$$

$$\Rightarrow f_{U_2}(u) = \frac{3}{2}(3-u)^2, \quad 2 \le u \le 4$$

(c). $$F_{U_3}(u) = \begin{cases} 0, & u < 0 \\ u^{\frac{3}{2}}, & 0 \le u \le 1 \\ 1, & u > 1 \end{cases}$$

$$\Rightarrow f_{U_3}(u) = \frac{3}{2} u^{\frac{1}{2}}, \quad 0 \le u \le 1$$

6.4 (a) $$F_U(u) = \begin{cases} 0, & u \le 1 \\ 1 - e^{-\frac{u-1}{12}}, & u > 1 \end{cases}$$

$$\Rightarrow f_U(u) = \frac{1}{12} e^{-\frac{u-1}{12}}, \quad u > 1$$

(b) $$E(U) = \int_1^{\infty} \frac{1}{12} \cdot u \cdot e^{-\frac{u-1}{12}} \, du$$

(let $z = u-1$) $$= \int_0^{\infty} \frac{1}{12}(z+1) e^{-\frac{z}{12}} \, dz.$$

$$= \int_0^{\infty} \frac{1}{12} z e^{-\frac{z}{12}} \, dz + \int_0^{\infty} \frac{1}{12} e^{-\frac{z}{12}} \, dz$$

$$= 12 + 1 = 13$$

6.6 (a) when $u \in [0, 1]$

$$P(U \le u) = P(Y_1 - Y_2 \le u)$$

$$= \int_0^u \int_{2y_2}^{y_2+u} 1 \; dy_1 \, dy_2$$

$$= \frac{u^2}{2}$$

when $u \in (1, 2]$.

$$P(U \le u) = P(Y_1 - Y_2 \le u)$$

$$= 1 - P(Y_1 - Y_2 > u)$$

$$= 1 - \int_u^2 \int_0^{y_1 - u} 1 \; dy_2 \, dy_1$$

$$= -\frac{u^2}{2} + 2u - 1$$

$$\Rightarrow F_U(u) = \begin{cases} 0 & , \; u < 0 \\ \frac{u^2}{2} & , \; 0 \le u \le 1 \\ 2u - \frac{u^2}{2} - 1 & , \; 1 < u \le 2 \\ 1 & , \; u > 2 \end{cases}$$

$$\Rightarrow f_U(u) = \begin{cases} u & , \; 0 \le u \le 1 \\ 2 - u & , \; 1 < u \le 2 \\ 0 & , \; o.w. \end{cases}$$

(b) $E(u) = \int_0^1 u \cdot u \, du + \int_1^2 u \cdot (2-u) \, du$

$= 1$

6.7. Skipped.

6.8 (a)

$$f_u(u) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} u^{\beta-1} (1-u)^{\alpha-1}, \quad 0 < u < 1$$

(b) $U \sim Beta(\beta, \alpha)$

(c) $$\left.\begin{aligned} E(u) &= \frac{\beta}{\alpha+\beta} \\ E(Y) &= \frac{\alpha}{\alpha+\beta} \end{aligned}\right\} \Rightarrow E(u) + E(Y) = 1$$

(d) $Var(u) = var(Y) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1}$

6.13 skipped.

6.14. First we know

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} 6y_1(1-y_1) \times 3y_2^2 , & 0 \le y_1, y_2 \le 1 \\ 0 , & o.w \end{cases}$$

$P(U \leq u)$

$= 1 - \int_u^1 \int_{u/y_2}^1 6y_1(1-y_1)\, 3y_2^2 \, dy_1 \, dy_2$

$= 9u^2 - 8u^3 + 6u^3 \ln u, \quad$ for $u \in [0,1]$

$\Rightarrow f_U(u) = 18u - 18u^2 + 18u^2 \ln u, \quad u \in [0,1]$

6.15 Skipped.

6.28 Let $U = -2\ln(Y)$ is a monotone function on $(0,1)$
The inverse is $y = h^{-1}(u) = e^{-u/2} \Rightarrow \frac{dy}{du} = -\frac{1}{2}e^{-\frac{u}{2}}$.

$f_U(u) = \frac{1}{2}e^{-u/2} \cdot 1 = \frac{1}{2}e^{-u/2}, \quad u > 0.$

6.29. 6.31
6.33. 6.35 Skipped.

6.63. 6.65 Skipped.

6.68 (a) Since $u = \frac{Y_1}{Y_2}$ and $u_2 = Y_2$, we have

$$\begin{cases} y_1 = u_1 u_2 \\ y_2 = u_2 \end{cases}. \quad \text{So } |J| = \begin{vmatrix} u_2 & u_1 \\ 0 & 1 \end{vmatrix} = u_2$$

$\Rightarrow f_{u_1 u_2}(u_1, u_2) = 8 u_1 u_2^3$.

Support of (u_1, u_2) : $\begin{cases} 0 \le u_1 < 1 \\ 0 < u_2 \le 1 \end{cases}$

$$\Rightarrow f_{u_1 u_2}(u_1, u_2) = \begin{cases} 8u_1 u_2^3, & \text{if } 0 \le u_1 < 1, 0 < u_2 \le 1 \\ 0, & \text{o.w} \end{cases}$$

(b) Joint pdf can be factored into a piece depending on u_1 only and a piece depending on u_2 only. $\Rightarrow$ u_1 and u_2 are independent.

6.70 Skipped.

6.60 $U \sim \chi_2^2$

6.46. $M_W(t)$

$= E(e^{tW}) = E(e^{t \cdot \frac{2Y}{\beta}})$

$= M_Y(\frac{2t}{\beta}) = \frac{1}{(1-\beta \cdot \frac{2t}{\beta})^\alpha} = \frac{1}{(1-2t)^\alpha} = \frac{1}{(1-2t)^{n/2}}$

$\Rightarrow W \sim \chi_n^2$

6.56 Let $Y_1 \sim exp(0.5)$, $Y_2 \sim exp(0.5)$ and $Y_1 \perp Y_2$.
By method of mgf,

$$U = Y_1 + Y_2 \sim Gamma(2, 0.5)$$

So $P(U > 1.5) = 1 - \int_0^{1.5} \frac{1}{\Gamma(2)\, 0.5^2} y^{2-1} e^{-y/0.5} dy$

$= \cdots$

$= .1991$

6.58 Skipped.